

Another comment on "A Lagrangian for DSR Particle and the Role of Noncommutativity"

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Abstract

Motivated by the discussion raised up in references [1,2,3], we wrote this work to complement [1,2,3], in the Doubly Special Relativity (DSR) framework. We show how the DSR particle dynamics described by a Lagrangian proposed in [1] can be understood without using the Dirac method for constrained systems or noncommutative structures. We are also concerned about the problem of describing DSR as Special Relativity in a different parameterization.

1 Introduction

In the last few years, a number of papers appeared about a generalization of Special Relativity (SR). The so called Doubly Special Relativity (DSR) [4,5,6,7] introduces a new observer independent scale, besides c , with similar properties to the latter. There is a number of attractive motivations for such a modification, listed in [7].

In order to introduce the new scale, the construction of DSR generalizes the conventional energy-momentum dispersion relation to $p^2 = m^2 + F(k, E)$, where m is the rest mass; F depends on the energy and on the new scale k , which may be related to the Planck mass. It is assumed that in the limit $k \rightarrow +\infty$, $F(k, E) \rightarrow 0$ and we recover the standard relation $p^2 = m^2$.

The most popular model, known as Magueijo-Smolín (MS) DSR [5,6], has the dispersion relation,

$$p^2 = m^2 \left(1 - \frac{p_0}{k}\right)^2. \quad (1)$$

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The MS construction is made in space of conserved energy-momentum $\{p_\mu\}$, which gives no space-time interpretation for such model [8].

In order to obtain a space-time picture of the model, a paper was written [1], that may correspond to MS DSR. The work [1] is separated in two parts:

- 1) The introduction of noncommutative phase space to maintain Lorentz invariance for DSR dispersion relation;
- 2) The issue of velocity of DSR particle described by the Lagrangian cited above.

We focus our attention on the second topic. Due to the choice of a non standard gauge, the definition of velocity $v^i = \frac{\dot{x}^i}{\dot{x}^0}$ [9] does not lead to the expected particle velocity in the $k \rightarrow +\infty$ limit.

In the next Section, we show that the particle dynamics of the proposed Lagrangian can be completely described without use of the Dirac method for constrained systems [10] and hence, no particular gauges need to be taken. We see that the expected results are found in the $k \rightarrow +\infty$ limit for the three velocity, with no change of initial variables, as in [1]. We also make a brief comment on the introduction of noncommutative structures. Section 3 is dedicated to complement the discussion raised in references [2,3]. In [2], the author insists that DSR physics behind the Lagrangian [1] is just the relativistic particle in a different parameterization. We confirm this fact and remind that all the DSR models are related with SR by means of a singular transformation. Hence, the question of equivalence turns out to be a rather delicate issue. Section 4 is left for conclusions.

2 Lagrangian dynamics of a DSR point particle

The suggestion of [1] is to consider the Lagrangian,

$$L = \frac{mk}{\sqrt{k^2 - m^2}} [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{m^2}{k^2 - m^2} (g_{\mu\nu} \dot{x}^\mu \eta^\nu)^2]^{\frac{1}{2}} - \frac{m^2 k}{k^2 - m^2} g_{\mu\nu} \dot{x}^\mu \eta^\nu. \quad (2)$$

$g_{\mu\nu}$ represents the metric in Minkowski space $g_{00} = -g_{ii} = 1$ and η^μ is a fixed four vector defined by $\eta^0 = 1$, $\eta^i = 0$. (We will use the following notation for contractions with the metric: $g_{\mu\nu} A^\mu B^\nu = (AB)$).

The conjugate momentum is,

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{mk}{\sqrt{k^2 - m^2}} \frac{\dot{x}_\mu + \frac{m^2}{k^2 - m^2}(\dot{x}\eta)\eta_\mu}{[\dot{x}^2 + \frac{m^2}{k^2 - m^2}(\dot{x}\eta)^2]^{\frac{1}{2}}} - \frac{m^2 k}{k^2 - m^2} \eta_\mu. \quad (3)$$

As noted in [1], it is easy to see that (3) satisfies the MS dispersion relation [5],

$$p^2 = m^2 \left[1 - \frac{(\eta p)}{k} \right]^2. \quad (4)$$

One notes that both the scale factor and the last term (total derivative) in (2) are required to achieve MS relation. We will use these facts to discuss the equivalence between (2) and the standard Lagrangian of the free relativistic particle in Section 3.

Since we wish to analyze the dynamics of the model described by the Lagrangian (2), let us find the equations of motion via Euler-Lagrangian equations,

$$\frac{\partial L}{\partial x^\mu} = \frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right). \quad (5)$$

After the first integration over time, one has,

$$\frac{mk}{\sqrt{k^2 - m^2}} \frac{\dot{x}_\mu + \frac{m^2}{k^2 - m^2}(\dot{x}\eta)\eta_\mu}{[\dot{x}^2 + \frac{m^2}{k^2 - m^2}(\dot{x}\eta)^2]^{\frac{1}{2}}} = \frac{m^2 k}{k^2 - m^2} \eta_\mu + p_\mu, \quad (6)$$

where p_μ is an arbitrary four vector.

We are interested in the kinematics of the model. Then, one takes, as natural definition for three-velocity, the following formula [9],

$$v^i \equiv \frac{\dot{x}^i}{\dot{x}^0}. \quad (7)$$

For this case, to find v^i it is sufficient to solve Eq. (6) only when μ assumes the values 1, 2, 3. Hence,

$$\frac{mk}{\sqrt{k^2 - m^2}} \frac{\dot{x}_i}{[\dot{x}^2 + \frac{m^2}{k^2 - m^2}(\dot{x}\eta)^2]^{\frac{1}{2}}} = p_i, \quad (8)$$

which gives us,

$$\dot{x}^i = \frac{kp^i}{\sqrt{m^2 k^2 + \vec{p}^2 (k^2 - m^2)}} \dot{x}^0. \quad (9)$$

The general solution of the differential equation (9) is given by,

$$x^i(x^0) = \frac{kp^i}{\sqrt{m^2k^2 + \vec{p}^2(k^2 - m^2)}}x^0 + A^i, \quad (10)$$

where A^i is an arbitrary three-vector and $i = 1, 2, 3$.

The above definition for physical three-velocity gives,

$$v^i = \frac{\dot{x}^i}{\dot{x}^0} = \frac{kp^i}{\sqrt{m^2k^2 + \vec{p}^2(k^2 - m^2)}}, \quad (11)$$

which has the magnitude of,

$$v = \sqrt{v^2} = \sqrt{\frac{k^2\vec{p}^2}{m^2k^2 + \vec{p}^2(k^2 - m^2)}}. \quad (12)$$

First of all, we note that v is exactly given by,

$$v = \frac{1}{V}, \quad (13)$$

where $V^i = \frac{\dot{x}^i}{\dot{x}^0}$ was defined in the equations (17) of [1]. It shows that the non standard gauge choice (see equation (9) in [1]),

$$\psi_1 = (xp) = 0, \quad (14)$$

enforces the definition of a new variable (see equation (18) in [1]),

$$X \equiv \left(\frac{k^2 - m^2}{k^2} + \frac{m^2}{\vec{p}^2}\right)\tilde{x}_0, \quad (15)$$

in order to obtain the free relativistic particle results when the $k \rightarrow +\infty$ limit is taken.

After definition of the X variable above, the three velocities of [1] (r.h.s. of Eq. (16)) and of this work (l.h.s. of Eq. (16)) coincide,

$$v = \frac{|\dot{\tilde{x}}^i|}{\dot{X}}. \quad (16)$$

Thus, conclusions are the same, namely,

1) $m^2 = 0 \Rightarrow v = 1$; this shows us that massless particles in the DSR framework also move at the speed of light c .

2) If we interpret \vec{p} as the three momentum of a DSR particle, then, $\vec{p}^2 = k^2 \Rightarrow v = 1$. Hence, massive particles can move with the speed of light, provided that \vec{p} has a magnitude of k .

3) $k \rightarrow +\infty \Rightarrow v = \sqrt{\frac{\vec{p}^2}{m^2 + \vec{p}^2}}$. This result is in accordance with the standard one, since $E^2 = m^2 + \vec{p}^2$ and $\vec{v} = \frac{\vec{p}}{E}$ for all particles [11].

We have shown that it is not necessary to use the Dirac method to analyze the dynamics of the model described. It avoids the non standard gauge choice which is accompanied by a change of initial variables in order to obtain expected results.

To finish this Section we comment the introduction of noncommutative structures by the author in [1].

First of all, the author observes that the Lorentz generator $J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$ fails to keep the MS dispersion relation invariant,

$$\{J_{\mu\nu}, (p^2 - m^2(1 - \frac{(\eta p)}{k})^2)\} = -\frac{2}{k}(1 - \frac{(\eta p)}{k})(\eta_\mu p_\nu - \eta_\nu p_\mu), \quad (17)$$

where $\{, \}$ is the Poisson bracket.

The resolution proposed was the introduction of a noncommutative phase space $\{\tilde{x}^\mu, p_\nu\}$, where the new Lorentz generator $\tilde{J}_{\mu\nu} = \tilde{x}_\mu p_\nu - \tilde{x}_\nu p_\mu$ will, in turn, keep the MS relation invariant. This is done by considering an appropriate Lagrangian, where the MS relation turns out to be a constraint $\psi_2 = p^2 - m^2(1 - \frac{(\eta p)}{k})^2$ presented in the correspondent Hamiltonian formulation.

Fixing the gauge $\psi_1 = (xp) = 0$, the Dirac brackets are defined in the following way,

$$\{A, B\}^* = \{A, B\} - \{A, \psi_i\}\{\psi_i, \psi_j\}^{-1}\{\psi_j, B\}, \quad (18)$$

where $\{\psi_i, \psi_j\}$ stands to the constraint matrix.

Considering a coordinate transformation,

$$\tilde{x}_\mu = x_\mu - \frac{1}{k}(x\eta)p_\mu, \quad (19)$$

one has the following noncommutative algebra,

$$\begin{aligned} \{\tilde{x}_\mu, \tilde{x}_\nu\}^* &= \frac{1}{k}(\tilde{x}_\mu \eta_\nu - \tilde{x}_\nu \eta_\mu) + \frac{k^2 - m^2}{k^2 m^2}(\tilde{x}_\mu p_\nu - \tilde{x}_\nu p_\mu), \\ \{\tilde{x}_\mu, p_\nu\}^* &= -g_{\mu\nu} + \frac{1}{k}(p_\mu \eta_\nu - p_\nu \eta_\mu) + \frac{k^2 - m^2}{k^2 m^2}p_\mu p_\nu, \end{aligned}$$

$$\{p_\mu, p_\nu\}^* = 0. \quad (20)$$

With the new phase space algebra and with $\tilde{J}_{\mu\nu} = \tilde{x}_\mu p_\nu - \tilde{x}_\nu p_\mu$, we see that,

$$\{\tilde{J}_{\mu\nu}, (p^2 - m^2(1 - \frac{(\eta p)}{k})^2)\}^* \approx 0, \quad (21)$$

and then the author in [1] advocates that MS relation is Lorentz invariant on-shell.

We remind that, for any function $A(x, p)$ defined in phase space,

$$\{A, \psi\}^* \approx 0, \quad (22)$$

where $\psi(x, p)$ is a constraint presented in the formulation. Hence, the motivation presented by [1] has no relation to the problem of keeping MS relation invariant.

3 DSR as SR in a different parameterization

In despite of all discussions we have made so far, we would like to ask the question: is the model proposed by Lagrangian (2) really the DSR model or does it represent the SR model in a different parameterization?

One way to answer this question was shown in [2], where the author discusses, in particular, the physics behind the Lagrangian (2).

It is shown in details in [2] how to find the Lagrangian (2) starting from the Lagrangian of free relativistic particle. This is accomplished by the redefinition of momenta $P_\mu \rightarrow f_\mu(p)$ (f_μ is an arbitrary invertible function), which can always be completed to a canonical transformation [12]. Thus, one can say, in accordance with [2], that the Lagrangian (2) describes the free relativistic particle in a different parameterization.

We confirm this fact in another way.

Consider the Lagrangian,

$$L = m(\dot{X}^\mu \dot{X}_\mu)^{\frac{1}{2}}, \quad (23)$$

which describes the free relativistic particle.

One takes the following invertible change of variables,

$$X^\mu \rightarrow x^\mu = M^\mu{}_\nu X^\nu, \quad (24)$$

where the only non vanishing terms of M are given by: $M^0_0 = \frac{k^2-m^2}{k^2}$ and $M^i_i = \frac{\sqrt{k^2-m^2}}{k}$; $k \neq m$.

In terms of the new coordinates, the Lagrangian (23) reads,

$$\begin{aligned} L(x, \dot{x}) &= m \left(\left(\frac{k^2}{k^2 - m^2} \right)^2 (\dot{x}^0)^2 - \frac{k^2}{k^2 - m^2} (\dot{x}^i)^2 \right)^{\frac{1}{2}} \\ &= \frac{mk}{\sqrt{k^2 - m^2}} (\dot{x}^\mu \dot{x}_\mu + \frac{m^2}{k^2 - m^2} (\eta^\mu \dot{x}_\mu)^2)^{\frac{1}{2}}. \end{aligned} \quad (25)$$

Two Lagrangians differing by a total derivative term are equivalent. Immediately one sees that Lagrangians (2) and (25) differ only by the term $\frac{m^2 k}{k^2 - m^2} g_{\mu\nu} \dot{x}^\mu \eta^\nu$, which is a total derivative. Thus one concludes that the Lagrangian (2) is equivalent to the Lagrangian (23) of free relativistic particle, as stated.

4 Conclusion

We have shown in Section 2 that particle dynamics of the Lagrangian (2) can be completely described without the use of the Dirac method for constrained systems [10] and hence, with no particular gauges. The expected results are found in the $k \rightarrow +\infty$ limit for the three velocity, with no change of initial variables, as in [1]. We also saw that the noncommutative phase space introduced by [1] does not solve the problem of keeping the MS relation invariant.

In Section 3 we confirmed the result raised in the discussion of the authors in [1,2,3]: the Lagrangian (2), that should describe a particle in the MS DSR framework, is just the Lagrangian of the standard free relativistic particle. We remind, then, that the problem of consistent construction of DSR models in configuration space is still an open issue. We remark that new ideas appear in order to solve this problem [13,14].

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References

- [1] Subir Ghosh, Phys.Rev. D74 (2006) 084019; hep-th/0608206.
- [2] J. Antonio García, hep-th/0705.0143.
- [3] Subir Ghosh, hep-th/0705.0586.
- [4] Giovanni Amelino-Camelia, Nature 418 (2002) 34-35; gr-qc/0207049.
- [5] J. Magueijo and L. Smolin, Phys. Rev. Lett. 88 (2002) 190403.
- [6] J. Magueijo and L. Smolin, Phys. Rev. D67 (2003) 044017; gr-qc/0207085.
- [7] A. A. Deriglazov and B. F. Rizzuti, Phys. Rev. D71 (2005) 123515; hep-th/0410087.
- [8] S. Mignemi, Phys.Rev. D68 (2003) 065029; gr-qc/0304029.
- [9] M. Daszkiewicz, K. Imilkowska and J. Kowalski-Glikman, Phys. Lett. A323 (2004) 345; hep-th/0304027
- [10] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva Univ., New York, 1964).
- [11] S. Weinberg, *Gravitation and Cosmology* (Willey, New York, 1972).
- [12] H. Goldstein, *Classical Mechanics* (Addison-Wesley, 1950).
- [13] S. Mignemi, gr-qc/0704.1728.
- [14] F. Girelli, S. Liberati and L. Sindoni, Phys. Rev D75, 064015 (2007); gr-qc/0611024.